

HIGHER DERIVATIVE QUANTUM GRAVITY WITH GAUSS-BONNET TERM

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ABSTRACT

Higher derivative theory is one of the important models of quantum gravity, renormalizable and asymptotically free within the standard perturbative approach. We consider the $4 - \epsilon$ renormalization group for this theory, an approach which proved fruitful in $2 - \epsilon$ models. A consistent formulation in dimension $n = 4 - \epsilon$ requires taking quantum effects of the topological term into account, hence we perform calculation which is more general than the ones done before. In the special $n = 4$ case we confirm a known result by Fradkin-Tseytlin and Avramidi-Barvinsky, while contributions from topological term do cancel. In the more general case of $4 - \epsilon$ renormalization group equations there is an extensive ambiguity related to gauge-fixing dependence. As a result, physical interpretation of these equations is not universal unlike we treat ϵ as a small parameter. In the sector of essential couplings one can find a number of new fixed points, some of them have no analogs in the $n = 4$ case.

Keywords: Higher Derivative Quantum Gravity, One-loop Divergences, Gauss-Bonnet term, Renormalization Group.

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1 Introduction

Quantizing gravitational field is an important aspect of quantum description of nature. Besides main efforts in the last decades were concentrated on the study of (super)string theories, recently there was an extensive development of a proper quantum gravity: both perturbative [1, 2, 3, 4] and non-perturbative [5, 6, 7, 8]. It is well known that the perturbative approach meets serious difficulty which can be summarized as a conflict between renormalizability and unitarity [5]. Quantum gravity based on General Relativity is non-renormalizable, in particular higher derivative divergences emerge starting already from the one-loop level [1, 2]. In the presence of matter fields or at second loop [9] the divergences persist on-shell, indicating that no “magic” cancellations occur. The situation in supergravity extensions of General Relativity is somehow better, because higher derivative divergences emerge only at higher loops [10]. However, from the

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general perspective there is no principal difference, for the corresponding version of supergravity extensions of General relativity are also non-renormalizable by power counting.

An alternative way of quantizing gravity is to introduce some higher derivative terms into a classical action, treating them at the same footing as the lower-derivative (Einstein-Hilbert and cosmological) terms. For example, adding generic fourth order derivative terms, one modifies propagator and vertices in such a way that the new quantum theory is renormalizable [3] (see also [11] for a more general formal proof of renormalizability). This nice property enables one to establish the asymptotic freedom in the UV limit [12, 13, 4, 14] and explore the possible role of quantum gravity in the asymptotic behavior for GUT-like models [15]. Introducing new terms with derivatives higher than four leads to super-renormalizable theories of quantum gravity [16].

Unfortunately, the price for renormalizability is very high. The propagator of renormalizable quantum gravity contains unphysical poles corresponding to the states with negative kinetic energy or with a negative norm in the state space. These unphysical states are conventionally called massive ghosts [3]³. Despite the ghost masses have magnitude of the Planck order, they spoil unitarity of the S-matrix exactly in the range of energies where quantum gravity is supposed to apply⁴. Indeed the presence of massive unphysical ghosts in the tree-level propagator does not necessarily imply that S-matrix is not unitary if one takes quantum corrections into account. It has been readily noticed [13, 23] that the quantum corrections to graviton propagator may, in principle, transform the real unphysical pole into a pair of complex poles. In this case the massive states should disappear in the asymptotic states and hence unitarity of the S-matrix gets restored. Unfortunately, until now there is no safe method to check whether this is really the effect of a resummation in the perturbative expansion. For example, despite the leading order of $1/N$ expansion provides corrections in appropriate form [23, 24], this approximation is not consistent in the case of quantum gravity [25]. Qualitatively the same is also true for other attempts in this direction and the final answer concerning the presence of ghosts in a full quantum theory remains unclear. In generic higher derivative gravitational theory (polynomial in derivatives) we meet complicated structure of the propagator, but with inevitable appearance of unphysical massive ghosts [16].

In order to clarify the situation with massive ghosts one needs to perform a non-perturbative treatment of the theory. In quantum theory, one of the known non-perturbative methods is the consideration of theory in a dimension $4 - \epsilon$, where ϵ is a small parameter. This method has been developed by Wilson and Fisher for investigating critical phenomena [26] and later on applied to quantum field theory problems [27]. In quantum gravity the same idea has been consistently applied only in two-dimensional models [28, 29, 5, 30, 31], where it proved very fruitful. The main lesson of the two-dimensional case is the correspondence between $2 + \epsilon$ approach and numerical non-perturbative methods, such as dynamical triangulations [6]. One can accept this

³General introduction to the fourth order derivative quantum gravity can be found in [17]. A detailed investigation of propagator in higher derivative gravity models has been recently performed in [18] (see also further references therein).

⁴There are interesting examples of higher derivative gravity theory with torsion which are free of ghosts [19, 20], but renormalizability can not be achieved in these models [21, 22].

is a hint indicating the potential importance of the $4 - \epsilon$ consideration. In the present paper we shall consider the $4 - \epsilon$ approach for higher derivative quantum gravity. In particular, it looks interesting to explore the possibility of new fixed points in $4 - \epsilon$ theory. The important aspect is that the Gauss-Bonnet term, which is a part of higher derivative gravity action, is quite relevant for this purpose. The reason is that, for the theory in $n = 4 - \epsilon$ dimensions, the Gauss-Bonnet term is not a topological term, in particular it contributes to many vertices and may be, in principle, affecting the β -functions in the $4 - \epsilon$ case.

The next question is whether the effect of the Gauss-Bonnet term has to vanish in $n \rightarrow 4$ limit. In fact, the opposite situation can not be ruled out beforehand [32]. The point is that the topological nature of this term is closely related to its diffeomorphism invariance. However, when theory is quantized (e.g., through the Faddeev-Popov procedure), the covariance is broken such that the vector space extends beyond physical degrees of freedom. After quantization, not only spin-2, but also spin-1 and spin-0 components of quantum metric become relevant, and the topological term creates new vertices of interaction between these components [32]. The quantum-gravitational loops may be, in principle, affected by the presence of topological term. Of course, this output does not look probable because, after including the topological term, the gauge-fixing condition should modify and eventually compensate the new vertices. But this is a sort of believes which are always good to verify and we will perform such verification in what follows. In a recent paper [33] we performed similar calculation for the particular case of Weyl quantum gravity and found that, in this case, the effect of Gauss-Bonnet term is relevant for $4 - \epsilon$ renormalization group but vanish when $n \rightarrow 4$. An extra benefit of the calculational scheme taking the Gauss-Bonnet term into account is the highest degree of safety for correctness of calculations. In case of Weyl theory we have confirmed the previous results [4, 34] for the $n = 4$ renormalization group and also established conformal invariance of one-loop counterterms. Here we shall generalize this calculation for generic non-conformal theory and present a more complete investigation of $4 - \epsilon$ renormalization group, taking into account a gauge-fixing dependence and the corresponding ambiguity.

In order to complete the picture and emphasize even more the relevance of the problem, let us remember that the Gauss-Bonnet-like term is an important ingredient of an effective low-energy action of the (super)string theory. The string effective action is composed by the Einstein-Hilbert term and an infinite set of higher derivative terms which emerge in the next order in α' (see, e.g., [35]). Therefore, models like effective low-energy quantum gravity [36, 37] which are supposed to be a kind of some universal low energy limit of string theories, may be sensitive to the presence of Gauss-Bonnet term. The uniqueness of the output in effective approaches depends on whether the $n \rightarrow 4$ limit is universal or not. Hence in case we find vanishing contribution from the Gauss-Bonnet term in $n \rightarrow 4$ limit, this will be a certain confirmation of universality of quantum corrections.

Thus, the main purpose of our work is to derive the one-loop corrections for higher-derivative quantum gravity with the Gauss-Bonnet term in $n \neq 4$ and analyze the consequent renormalization group. The paper is organized as follows. In the next section we introduce notations and briefly discuss the theory. Section 3 contains brief summary of a Lagrangian quantization

of the theory and in particular the discussion of gauge-fixing conditions and related arbitrariness in counterterms. In section 4 we derive the trace of the coincidence limit of second Schwinger-DeWitt coefficient for an arbitrary dimension n . Furthermore, we consider the $n \rightarrow 4$ limit and obtain the expression for the divergences. The cancellation of contributions from topological term and perfect fitting with the previous calculation of divergences in the higher derivative quantum gravity [14] confirm the correctness of our calculation. In section 5 we perform analytical and numerical investigation of $4 - \epsilon$ renormalization group equations for the higher derivative couplings and in section 6 consider the renormalization group equations for the Newton and cosmological constants. Finally, in section 7 we draw our conclusions. Many technical details and bulky formulas are collected in Appendices. Throughout the paper we use Euclidean signature in order to be consistent with the previous publications [4, 14].

2 General framework and notations

The classical action of the theory has the form

$$S = -\mu^{n-4} \int d^n x \sqrt{g} \left\{ \frac{1}{2\lambda} C^2 - \frac{1}{\rho} E + \frac{1}{\xi} R^2 + \tau \square R - \frac{1}{\kappa^2} (R - 2\Lambda) \right\}, \quad (1)$$

where μ is some dimensional parameter,

$$C^2 = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - \frac{4}{n-2} R_{\alpha\beta} R^{\alpha\beta} + \frac{2}{(n-1)(n-2)} R^2$$

is the square of Weyl tensor,

$$E = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2$$

is the integrand of the Gauss-Bonnet term, which is topological in $n = 4$ and λ, ρ, ξ, τ are independent parameters in the higher derivative sector of the action. Furthermore, $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$ and $\kappa^2 = 16\pi G$, where G is the Newton constant and Λ is the cosmological constant.

An alternative form for the action (1) is

$$S = -\mu^{(n-4)} \int d^n x \sqrt{g} \left\{ x R_{\mu\nu\alpha\beta}^2 + y R_{\mu\nu}^2 + z R^2 + \tau \square R - \frac{1}{\kappa^2} (R - 2\Lambda) \right\}, \quad (2)$$

where we denote $R_{\mu\nu\alpha\beta}^2 = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$ and $R_{\alpha\beta}^2 = R_{\alpha\beta} R^{\alpha\beta}$. The parameters x, y, z are related to ρ, λ, ξ as follows:

$$x = \frac{1}{2\lambda} - \frac{1}{\rho}, \quad y = -\frac{2}{(n-2)\lambda} + \frac{4}{\rho}, \quad z = -\frac{1}{\rho} + \frac{1}{\xi} + \frac{1}{\lambda(n-1)(n-2)}. \quad (3)$$

The two forms of the action (1) and (2) are completely equivalent and we shall use one or another depending on convenience.

At the classical level in the four dimensional space the role of the Gauss-Bonnet $\int E$ -term is negligible. However, situation changes when theory is considered in $n \neq 4$, where the $\int E$ -term becomes dynamical and hence must be taken into account. Using the language of Feynman diagrams, one can say that despite this term does not contribute to graviton propagator on flat

background for any n , it does contribute to vertices of interaction between different modes of metric (spin-2, spin-1 and spin-0). Hence the trivial nature of this term must be taken with proper caution, especially in $n \neq 4$. Let us mention that, for $n = 4 - \epsilon$, there is a natural limit $\epsilon = 1$ imposed by the following circumstance. For $n = 3$ the expressions for E and C^2 coincide, and since the Weyl tensor vanishes at $n = 3$, the Gauss-Bonnet term also vanishes.

At quantum level, all terms in the action (1) are necessary for renormalizability. At the same time, the term C^2 has a key role. If we do not include this term, the quantum theory will have, in the graviton (traceless and transverse) sector of quantum metric the situation when the propagator has only second derivatives while there are four derivative vertices. Then the standard evaluation of the superficial degree of divergences for the diagrams of the theory (see, e.g. [17]) shows very bad renormalizability properties which are even worse than the ones for the quantum gravity based on General Relativity. On the other hand, since the term C^2 is included, the propagator has, in the momentum representation, the form

$$G \sim \frac{1}{k^2(k^2 + m^2)} = \frac{1}{m_2^2} \left(\frac{1}{k^2} - \frac{1}{k^2 + m_2^2} \right), \quad (4)$$

where the negative sign of the second term indicates the ghost nature of the propagating spin-2 state with mass m_2 . Other unphysical mode is possible in the spin-0 sector of metric, depending on the coefficient of the $\int R^2$ -term [3, 17, 18].

Without R^2 -term, the higher derivative sector of theory possesses (in $n = 4$ case) local conformal invariance and its renormalization has some special features. The main point is that conformal symmetry is violated by anomaly, therefore this symmetry is not supposed to hold in renormalization beyond one-loop level. We have recently discussed renormalization in conformal quantum gravity in [33] and hence will not consider it here.

The relevance of the Gauss-Bonnet term at quantum level is also significant. Long ago has been noticed that the topological nature of this term is related to Bianchi identity [32]. As we already mentioned in the Introduction, it is in principle possible that the loop contributions depend on the presence of topological term. In particular, there is no reason to expect that the effect of Gauss-Bonnet term is negligible for $n = 4 - \epsilon$ -dimension renormalization group, for this term is not topological at $n \neq 4$ even at classical level. In previous paper [33] we have confirmed these considerations for the special case of Weyl quantum gravity and will now generalize these results for general higher derivative quantum gravity (1).

3 Quantization and gauge fixing

The general scheme of Lagrangian quantization for theory (1) has been formulated in [3] (see also [17] for more detailed exposition). The most useful method of practical calculations is through the background fields method and Schwinger-DeWitt technique (see [38] for an extensive introduction including the higher derivative case). The application of these methods to the case of higher derivative gravity has some peculiarities which have been revealed in [4] (see also [39, 17] for more detailed pedagogical consideration).

The background field method implies special parametrization of the metric

$$g_{\mu\nu} \longrightarrow g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}. \quad (5)$$

In the *r.h.s.* of the last formula, $g_{\mu\nu}$ is a background metric and $h_{\mu\nu}$ is a quantum field (integration variable in the path integral). The 1-loop effective action $\bar{\Gamma}^{(1)}$ can be written as [4]

$$\bar{\Gamma}^{(1)}[g_{\mu\nu}] = \frac{i}{2} \ln \text{Det } \hat{\mathcal{H}} - \frac{i}{2} \ln \text{Det } Y^{\alpha\beta} - i \ln \text{Det } \hat{\mathcal{H}}_{gh}, \quad (6)$$

where $\hat{\mathcal{H}}$ is bilinear (in the quantum fields) form of the action (1), taken together with the gauge-fixing term

$$S_{GF} = \mu^{n-4} \int d^n x \sqrt{g} \chi_\alpha Y^{\alpha\beta} \chi_\beta. \quad (7)$$

$\hat{\mathcal{H}}_{gh}$ is bilinear form of the action of the Faddeev-Popov ghosts, μ is the renormalization parameter in dimensional regularization. The expression (6) includes also the contribution of the weight function $Y^{\alpha\beta}$. In the case of higher derivative quantum gravity theory this term gives relevant contribution to the effective action, because $Y^{\alpha\beta}$ is a second order differential operator [4].

The general form of the gauge fixing conditions (here we restrict our attention to linear background gauges) has the form

$$\begin{aligned} \chi^\mu &= \nabla_\lambda h^{\lambda\mu} + \beta \nabla^\mu h \\ Y_{\mu\nu} &= \frac{1}{\alpha} \left(g_{\mu\nu} \square + \gamma \nabla_\mu \nabla_\nu - \delta \nabla_\nu \nabla_\mu + p_1 R_{\mu\nu} + p_2 R g_{\mu\nu} \right), \end{aligned} \quad (8)$$

where $\alpha_i = (\alpha, \beta, \gamma, \delta, p_1, p_2)$ are arbitrary gauge-fixing parameters. The action of the Faddeev-Popov ghosts can be written as

$$S_{gh} = \mu^{n-4} \int d^n x \sqrt{g} \bar{C}^\mu (\mathcal{H}_{gh})_\mu^\nu C_\nu, \quad (9)$$

where

$$\hat{\mathcal{H}}_{gh} = (\mathcal{H}_{gh})_\mu^\nu = -\delta_\mu^\nu \square - \nabla^\nu \nabla_\mu - 2\beta \nabla_\mu \nabla^\nu. \quad (10)$$

Generally speaking, the counterterms may depend on the choice of gauge-fixing parameters, but this dependence is constrained by the on-shell gauge-fixing independence. Let us consider these restrictions, generalizing similar discussion of [40] for the general n case.

We denote $\Gamma(\alpha_i)$ the effective action corresponding to arbitrary values of gauge parameters α_i and $\Gamma_m = \Gamma(\alpha_i^{(0)})$ calculated for special values of these parameters, $\alpha_i^{(0)}$. Our purpose is to evaluate the gauge fixing dependence $\Gamma(\alpha_i)$ or, equivalently, the expression for the difference between the two effective actions $\Gamma(\alpha_i) - \Gamma_m$. In general, this expression may be rather complicated (see the details in [40]), but the part of $\Gamma(\alpha_i) - \Gamma_m$ which is relevant for the renormalization group may be easily evaluated without special calculations. For this end we remember that the gauge dependence of counterterms has to disappear on the classical mass-shell (see, e.g., [41] for a general proof for gauge theories). Hence we can write

$$\Gamma(\alpha_i) = \Gamma_m + \int d^n x \sqrt{g} f_{\mu\nu}(\alpha_i) \frac{\delta S}{\delta g_{\mu\nu}}, \quad (11)$$

where $f_{\mu\nu}(\alpha_i)$ is some unknown function. The integration is taken over n -dimensional space, because our target is the renormalization group in $n = 4 - \epsilon$ dimensions.

Since the object of interest is the divergent part of the effective action $\Gamma(\alpha_i)$, one can assume it is a local expression. Furthermore, both Γ_m and $\Gamma(\alpha_i)$ have the same dimension as the classical equations of motion $\delta S/\delta g_{\mu\nu}$. Therefore $f_{\mu\nu}(\alpha_i)$ is a symmetric dimensionless tensor and the unique choice for it is

$$f_{\mu\nu}(\alpha_i) = g_{\mu\nu} \times f(\alpha_i), \quad (12)$$

where $f(\alpha_i)$ is a numerical quantity. Thus we arrive at the relation

$$\Gamma(\alpha_i) = \Gamma_m + f(\alpha_i) \times \int d^n x \sqrt{g} g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}}. \quad (13)$$

According to the last relation, the gauge-fixing dependence of divergent part (or, more general, the local part) of one-loop effective action is proportional to the trace of classical equations of motion. The corresponding traces for the relevant terms have the form

$$\begin{aligned} g_{\rho\sigma} \frac{\delta}{\delta g_{\rho\sigma}} \int d^4 x \sqrt{g} R_{\rho\sigma\lambda\theta}^2 &= \sqrt{g} \left\{ \frac{n-4}{2} R_{\rho\sigma\lambda\theta}^2 - 2 \square R \right\}, \\ g_{\rho\sigma} \frac{\delta}{\delta g_{\rho\sigma}} \int d^4 x \sqrt{g} R_{\rho\sigma}^2 &= \sqrt{g} \left\{ \frac{n-4}{2} R_{\rho\sigma}^2 - \frac{n}{2} \square R \right\}, \\ g_{\rho\sigma} \frac{\delta}{\delta g_{\rho\sigma}} \int d^4 x \sqrt{g} R^2 &= \sqrt{g} \left\{ \frac{n-4}{2} R^2 - 2(n-1) \square R \right\}, \\ g_{\rho\sigma} \frac{\delta}{\delta g_{\rho\sigma}} \int d^4 x \sqrt{g} (R - 2\Lambda) &= \sqrt{g} \left\{ \frac{n-2}{2} R - n \Lambda \right\}. \end{aligned} \quad (14)$$

At this point it is better to separate the $n = 4$ and $n = 4 - \epsilon$ cases. For $n = 4$ we obtain the following formula for the gauge-fixing dependence [40]:

$$\Gamma(\alpha_i) = \Gamma_m + f(\alpha_i) \times \int d^4 x \sqrt{g} \left\{ -\frac{1}{\kappa^2} (R - 4\Lambda) + \frac{6}{\xi} \square R \right\}. \quad (15)$$

According to (15), the coefficients of E , C^2 and R^2 poles are gauge-fixing invariant. One of the consequences is that the corresponding renormalization group equations are universal, providing information about the UV limit of the theory. At the same time, $(\square R)$ -type pole depends on the choice of the gauge-fixing condition. Taking different values of gauge-fixing parameters α_i one can provide any desirable value of the function $f(\alpha_i)$ and hence change the $(\square R)$ -type counterterm. Thus, the corresponding parameter τ in (1) is not essential. The immediate conclusion is that there is no much interest to calculate the renormalization of τ , especially if the calculation is done for a particular gauge-fixing. Another situation takes place for the Einstein-Hilbert and cosmological terms. Renormalization of each of them is gauge-fixing dependent, but one can easily check that the dimensionless combination $\kappa^2 \Lambda$ is an essential coupling constant with the invariant renormalization relation.

In the $n = 4 - \epsilon$ case the situation is more complicated. Simple calculation using (14) yields the following generalization of (15):

$$\Gamma(\alpha_i) = \Gamma_m + f(\alpha_i) \times \mu^{n-4} \int d^n x \sqrt{g} \left\{ \left[2x + \frac{ny}{2} + 2(n-1)z \right] (\square R) \right.$$

$$+ \frac{n-4}{2} (x R_{\mu\nu\alpha\beta}^2 + y R_{\mu\nu}^2 + z R^2) - \frac{n-2}{2\kappa^2} R + \frac{n\Lambda}{\kappa^2} \Big\}. \quad (16)$$

An important consequence of the eq. (16) is that neither one of the parameters $x, y, z, \tau, \kappa, \Lambda$ is essential in the $n = 4 - \epsilon$ case. However, gauge-fixing dependence is concentrated in a single numerical function $f(\alpha_i)$ and therefore we can easily extract combinations of the couplings which are essential parameters. This result will be extensively used in sections 5 and 6, where we consider renormalization group equations for essential couplings.

4 Bilinear expansion and derivation of divergences

In order to apply the background field method, we need the bilinear expansion in $h_{\mu\nu}$ for the classical action (1). It proves useful to work with the equivalent form of the action (2), and therefore we expand this action as follows:

$$\begin{aligned} S^{(2)} &= -\mu^{(n-4)} \int d^n x \left\{ x [\sqrt{g} R_{\mu\nu\alpha\beta}^2]^{(2)} + y [\sqrt{g} R_{\mu\nu}^2]^{(2)} + z [\sqrt{g} R^2]^{(2)} \right. \\ &\quad \left. - \frac{1}{\kappa^2} [\sqrt{g} R - 2\sqrt{g} \Lambda]^{(2)} \right\}. \end{aligned} \quad (17)$$

The expressions $[\sqrt{g} R_{\mu\nu\alpha\beta}^2]^{(2)}$, $[\sqrt{g} R_{\mu\nu}^2]^{(2)}$, $[\sqrt{g} R^2]^{(2)}$, $[\sqrt{g} R]^{(2)}$, $[\sqrt{g} \Lambda]^{(2)}$ were derived in the previous work [33] and we will not reproduce them here⁵. Using these formulas and the expression for the gauge-fixing term (7), after performing some cumbersome commutations [33] one can find the bilinear form of the action

$$[S + S_{GF}]^{(2)} = h^{\mu\nu} \mathcal{H}_{\mu\nu, \alpha\beta} h^{\alpha\beta}. \quad (18)$$

The operator $\hat{\mathcal{H}} = \mathcal{H}_{\mu\nu, \alpha\beta}$ depends on gauge parameters, $\alpha, \beta, \gamma, \delta, p_1$ and p_2 . For practical calculations these parameters have to be chosen in such a way that \mathcal{H} assumes the most simple (minimal) form

$$\hat{\mathcal{H}} = \hat{K} \square^2 + \mathcal{O}(\nabla^2), \quad (19)$$

where \hat{K} is a non-degenerate c -number operator. The expressions for the gauge-fixing parameters providing cancellation of the non-minimal four derivative terms $g_{\mu\nu} \nabla_\alpha \square \nabla_\beta$, $g_{\alpha\beta} \nabla_\mu \square \nabla_\nu$, $\nabla_\mu \nabla_\nu \nabla_\alpha \nabla_\beta$ and $g_{\nu\beta} \nabla_\mu \square \nabla_\alpha$ are the following:

$$\beta = \frac{y+4z}{4(x-z)}, \quad \alpha = \frac{2}{y+4x}, \quad \gamma = \frac{2x-2z}{y+4x}, \quad \delta = 1, \quad p_1 = p_2 = 0. \quad (20)$$

With this choice of the gauge parameters, the operator $\hat{\mathcal{H}}$ takes the form (19)

$$\hat{\mathcal{H}} = \hat{K} \square^2 + \hat{D}^{\rho\lambda} \nabla_\rho \nabla_\lambda + \hat{N}^\lambda \nabla_\lambda - (\nabla_\lambda \hat{Z}^\lambda) + \hat{W}, \quad (21)$$

where \hat{K} , $\hat{D}^{\rho\lambda}$, \hat{N}^λ , \hat{Z}^λ and \hat{W} are local matrix expressions in the $h^{\mu\nu}$ -space. The identity matrix in this space is defined as a symmetric tensor

$$\hat{1} = \delta_{\mu\nu, \alpha\beta} = \frac{1}{2} (g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha}). \quad (22)$$

⁵All but the first expansions can be also found in [12, 29, 14, 17].

Before writing down the bulky formulas for the elements of (21), let us notice that the expressions for \hat{N}^λ and \hat{Z}^λ are in fact irrelevant. The reason is that both of them are proportional to the covariant derivatives of the curvature tensor. Therefore, due to the locality of divergences, these terms may contribute only to irrelevant gauge-fixing dependent $\int \sqrt{g} \square R$ counterterm. Since we are not calculating this term here, for the sake of simplicity, in what follows we shall simply set both \hat{N}^λ and \hat{Z}^λ to zero.

The matrix coefficient \hat{K} of the fourth derivative term has the following form:

$$(\hat{K})_{\mu\nu,\alpha\beta} = \frac{y+4x}{4} \left[\delta_{\mu\nu,\alpha\beta} + \frac{y+4z}{4(x-z)} g_{\mu\nu} g_{\alpha\beta} \right]. \quad (23)$$

The expressions for $\hat{D}^{\rho\lambda}$ and \hat{W} have the form

$$\begin{aligned} (\hat{D}^{\rho\lambda})_{\mu\nu,\alpha\beta} = & -2x g_{\nu\beta} R_\alpha^{\rho\lambda}{}_\mu + 4x \delta_\nu^\rho R^\lambda{}_{\alpha\mu\beta} + (3x+y) g^{\rho\lambda} R_{\mu\alpha\nu\beta} \\ & - (4x+2y) \delta_\alpha^\rho g_{\nu\beta} R^\lambda{}_\mu - 2x g_{\nu\beta} R_{\mu\alpha} g^{\rho\lambda} + y g_{\mu\nu} \delta_\alpha^\rho R^\lambda{}_\beta \\ & - 2z \delta_{\alpha\beta}^{\rho\lambda} R_{\mu\nu} + 2z g_{\mu\nu} g^{\rho\lambda} R_{\alpha\beta} + 2x g_{\mu\nu} R_\alpha^{\rho\lambda}{}_\beta + 2x \delta_{\nu\beta}^{\rho\lambda} R_{\mu\alpha} \\ & + \left(\frac{z}{2} R - \frac{1}{4\kappa^2} \right) \left(\delta_{\mu\nu,\alpha\beta} g^{\rho\lambda} - g_{\mu\nu} g_{\alpha\beta} g^{\rho\lambda} - 2g_{\nu\beta} \delta_{\mu\alpha}^{\rho\lambda} + 2g_{\mu\nu} \delta_{\alpha\beta}^{\rho\lambda} \right) \\ & + \frac{y+2x}{2} \delta_{\mu\nu,\alpha\beta} R^{\rho\lambda} - \frac{y}{4} g_{\mu\nu} g_{\alpha\beta} R^{\rho\lambda}, \end{aligned} \quad (24)$$

$$\begin{aligned} (\hat{W})_{\mu\nu,\alpha\beta} = & \frac{3x}{2} g_{\nu\beta} R_{\mu\sigma\tau\theta} R_\alpha^{\sigma\tau\theta} + \frac{x-y}{2} R^\sigma{}_{\alpha\mu}{}^\tau R_{\nu\beta\sigma\tau} + \frac{5x+y}{2} R^\sigma{}_{\alpha\mu}{}^\tau R_{\sigma\nu\beta\tau} \\ & + \frac{3x+y}{2} R^\sigma{}_{\mu}{}^\tau R_{\sigma\alpha\tau\beta} + \frac{y-5x}{2} R_{\mu\tau} R^\tau{}_{\alpha\nu\beta} + \frac{y+2x}{2} R_{\mu\alpha} R_{\nu\beta} \\ & + \left(\frac{z}{2} R - \frac{1}{4\kappa^2} \right) (R_{\mu\alpha\nu\beta} - \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} + 3g_{\nu\beta} R_{\alpha\mu}) + z R_{\mu\nu} R_{\alpha\beta} \\ & - \frac{1}{4} \left[x R_{\rho\lambda\sigma\tau}^2 + y R_{\rho\lambda}^2 + z R^2 - \frac{1}{\kappa^2} (R - 2\Lambda) \right] \left(\delta_{\mu\nu,\alpha\beta} - \frac{1}{2} g_{\mu\nu} g_{\alpha\beta} \right) \\ & + \frac{3y}{2} g_{\nu\beta} R_{\mu\tau} R^\tau{}_\alpha - x g_{\mu\nu} R_{\alpha\theta\sigma\tau} R_\beta^{\theta\sigma\tau} - y g_{\mu\nu} R_{\alpha\tau} R^\tau{}_\beta. \end{aligned} \quad (25)$$

In the last two expressions we have used special condensed notations which help presenting them in a relatively compact way. All algebraic symmetries are implicit, including the symmetrization in the couples of indices $(\alpha\beta) \leftrightarrow (\mu\nu)$, $(\alpha \leftrightarrow \beta)$ and $(\mu \leftrightarrow \nu)$ and in the couple $(\rho \leftrightarrow \lambda)$ in the operator $\hat{D}^{\rho\lambda}$. In order to obtain the complete formula explicitly, one has to restore all these symmetries. For example,

$$R_{\mu\rho} R^\rho{}_{\alpha\nu\beta} \rightarrow \frac{1}{2} (R_{\mu\rho} R^\rho{}_{\alpha\nu\beta} + R_{\alpha\rho} R^\rho{}_{\mu\beta\nu})$$

restores the $(\alpha\beta) \leftrightarrow (\mu\nu)$ symmetry. Finally, let us notice that formulas (23), (24), (25) coincide with similar expressions of [14] (see the last reference therein) in the particular case $1/\rho \rightarrow 0$ after imposing the corresponding constraints on (x, y, z) in (3).

The solution for the inverse matrix \hat{K}^{-1} can be easily found in the form

$$\hat{K}^{-1} = (\hat{K}^{-1})_{\mu\nu}{}^{\theta\omega} = \frac{4}{4x+y} \left(\delta_{\mu\nu}{}^{\theta\omega} - \Omega g_{\mu\nu} g^{\theta\omega} \right),$$

where

$$\Omega = \frac{y + 4z}{4x - 4z + n(y + 4z)}.$$

Let us remark that in dimensional regularization the divergences of the expressions $\ln \text{Det } \hat{\mathcal{H}}$ and $\ln \text{Det } (K^{-1} \hat{\mathcal{H}})$ are the same, because an extra factor \hat{K}^{-1} is a c -number operator.

By straightforward algebra, we obtain the minimal operator in the standard form, useful for application of the Schwinger-DeWitt technique [4, 38]

$$\hat{H} = \hat{K}^{-1} \hat{\mathcal{H}} = \hat{1} \square^2 + \hat{V}^{\rho\lambda} \nabla_\rho \nabla_\lambda + \hat{U}. \quad (26)$$

It is important to notice that the new expressions

$$\hat{V}^{\rho\lambda} = \hat{K}^{-1} \hat{D}^{\rho\lambda} \quad \text{and} \quad \hat{U} = \hat{K}^{-1} \hat{W} \quad (27)$$

do not possess the symmetry $(\alpha\beta) \leftrightarrow (\mu\nu)$, while all other symmetries are preserved. The expressions for the two matrices $\hat{U} = (\hat{U})_{\mu\nu, \alpha\beta}$ and $\hat{V}^{\rho\lambda} = (\hat{V}^{\rho\lambda})_{\mu\nu, \alpha\beta}$ are very bulky and we settle them in Appendix A.

The algorithm for the one-loop divergences of a minimal fourth order operator (21) is the following [4] (see also [38] for an alternative, more systematic derivation):

$$\frac{1}{2} \ln \text{Det } \hat{H} \Big|_{\text{div}} = - \frac{\mu^{n-4}}{(4\pi)^2(n-4)} \int d^n x \sqrt{g} \text{tr} \lim_{x' \rightarrow x} a_2(x', x) = - \frac{\mu^{n-4}}{n-4} A_2, \quad (28)$$

where

$$\begin{aligned} \lim_{x' \rightarrow x} a_2(x', x) &= \frac{\hat{1}}{90} R_{\rho\lambda\kappa\omega}^2 - \frac{\hat{1}}{90} R_{\rho\lambda}^2 + \frac{\hat{1}}{36} R^2 + \frac{1}{6} \hat{\mathcal{R}}_{\rho\lambda} \hat{\mathcal{R}}^{\rho\lambda} - \hat{U} \\ &\quad - \frac{1}{6} R_{\rho\lambda} \hat{V}^{\rho\lambda} + \frac{1}{12} R \hat{V}^\rho{}_\rho + \frac{1}{48} \hat{V}^\rho{}_\rho \hat{V}^\lambda{}_\lambda + \frac{1}{24} \hat{V}_{\rho\lambda} \hat{V}^{\rho\lambda}. \end{aligned} \quad (29)$$

Here $\hat{\mathcal{R}}_{\rho\lambda}$ is the commutator of the covariant derivatives acting in the tensor $h^{\alpha\beta}$ space,

$$\hat{\mathcal{R}}_{\rho\lambda} = [\nabla_\rho, \nabla_\lambda]. \quad (30)$$

The particular traces of the expression (29) are collected in the Appendix B. The result must be summed up with the contributions of ghosts $\ln \text{Det } \hat{\mathcal{H}}_{gh}$ and the weight operator $\ln \text{Det } Y^{\alpha\beta}$, according to the formula (6). The corresponding expressions are also given in Appendix B. Let us present just the overall result for the divergent part of the effective action, in terms of parameters λ , ρ and ξ

$$\begin{aligned} A_2^t &= -\mu^{n-4} \int d^n x \sqrt{g} \left\{ \beta_1(n) E + \beta_2(n) C^2 + \beta_3(n) R^2 + \frac{\beta_4(n)}{\kappa^2} R \right. \\ &\quad \left. + \beta_5(n) \frac{\Lambda}{\kappa^2} + \frac{\beta_6(n)}{\kappa^4} \right\}. \end{aligned} \quad (31)$$

The coefficients (β -functions, as we shall see later on) are given by the expressions

$$\beta_i(n) = \frac{1}{(4\pi)^2} \left[\delta_i^{(0)} + \frac{\delta_i^{(1)}}{\rho} + \frac{\delta_i^{(2)}}{\rho^2} \right], \quad i = (1, 2, 3, 4, 5, 6). \quad (32)$$

For the sake of convenience let us now replace $n = 4 - \epsilon$, such that all coefficients $\delta_i^{(j)}(\xi, \lambda)$ should be expressed in terms of ϵ . The explicit form of the coefficients $\delta_l^{(j)}$ corresponding to β -functions for ρ , λ and ξ are collected in Appendix C. Their general dependence on the couplings has the following structure. For β_1 we have

$$\delta_1^{(0)} = \delta_{1A}^{(0)} + \delta_{1B}^{(0)} \frac{\lambda}{\xi}, \quad \delta_1^{(1)} = \delta_{1A}^{(1)} \xi + \delta_{1B}^{(1)} \lambda, \quad \delta_1^{(2)} = \delta_{1A}^{(2)} \xi^2 + \delta_{1B}^{(2)} \xi \lambda + \delta_{1C}^{(2)} \lambda^2. \quad (33)$$

For β_2 we have the expressions

$$\begin{aligned} \delta_2^{(0)} &= \delta_{2A}^{(0)} \frac{\xi}{\lambda} + \delta_{2B}^{(0)} + \delta_{2C}^{(0)} \frac{\lambda}{\xi}, & \delta_2^{(1)} &= \delta_{2A}^{(1)} \xi + \delta_{2B}^{(1)} \lambda, \\ \delta_2^{(2)} &= \delta_{2A}^{(2)} \xi^2 + \delta_{2B}^{(2)} \xi \lambda + \delta_{2C}^{(2)} \lambda^2. \end{aligned} \quad (34)$$

The coefficients of β_3 can be written as

$$\begin{aligned} \delta_3^{(0)} &= \delta_{3A}^{(0)} + \delta_{3B}^{(0)} \frac{\lambda}{\xi} + \delta_{3C}^{(0)} \frac{\lambda^2}{\xi^2}, & \delta_3^{(1)} &= \delta_{3A}^{(1)} \xi + \delta_{3B}^{(1)} \lambda + \delta_{3C}^{(1)} \frac{\lambda^2}{\xi}, \\ \delta_3^{(2)} &= \delta_{3A}^{(2)} \xi^2 + \delta_{3B}^{(2)} \xi \lambda + \delta_{3C}^{(2)} \lambda^2, \end{aligned} \quad (35)$$

Furthermore, the coefficients $\delta_4^{(j)}$ have the form

$$\delta_4^{(0)} = \delta_{4A}^{(0)} \xi + \delta_{4B}^{(0)} \lambda + \delta_{4C}^{(0)} \frac{\lambda^2}{\xi}, \quad \delta_4^{(1)} = \delta_{4A}^{(1)} \xi^2 + \delta_{4B}^{(1)} \lambda^2, \quad \delta_4^{(2)} = 0. \quad (36)$$

Finally, the coefficients of functions β_5 and β_6 have the form

$$\delta_5^{(0)} = \delta_{5A}^{(0)} \xi + \delta_{5B}^{(0)} \lambda, \quad \delta_6^{(0)} = \delta_{6A}^{(0)} \xi^2 + \delta_{6B}^{(0)} \lambda^2, \quad \delta_5^{(1,2)} = \delta_6^{(1,2)} = 0. \quad (37)$$

The coefficients $\delta_{iA,B,C}^{(k)}$ depend only on ϵ and are listed in Appendix C.

After taking the limit $n \rightarrow 4$ in coefficients $\beta_i(n)$, we arrive at the counterterm, that is the negative of the $n \rightarrow 4$ coefficient for the pole term

$$\begin{aligned} \Delta S = -\Gamma_{\text{div}}^{(1)} &= \frac{\mu^{n-4}}{(4\pi)^2(n-4)} \int d^n x \sqrt{g} \left\{ \frac{133}{20} C^2 - \frac{196}{45} E \right. \\ &+ \left(\frac{10 \lambda^2}{\xi^2} - \frac{5 \lambda}{\xi} + \frac{5}{36} \right) R^2 + \left(\frac{\xi}{12 \lambda} - \frac{13}{6} - \frac{10 \lambda}{\xi} \right) \frac{\lambda}{\kappa^2} R \\ &+ \left(\frac{56}{3} - \frac{2 \xi}{9 \lambda} \right) \frac{\lambda \Lambda}{\kappa^2} + \left(\frac{\xi^2}{72} + \frac{5 \lambda^2}{2} \right) \frac{1}{\kappa^4} \left. \right\}. \end{aligned} \quad (38)$$

The last expression does agree with the well known result of Avramidi and Barvinsky [14]⁶. This coincidence is remarkable in several aspects. First of all, one can see that the effect of the Gauss-Bonnet term is not relevant for one-loop renormalization. This means that, in the framework of fourth derivative quantum gravity, the hypothesis of Capper and Kimber [32] concerning the relevance of the topological term on quantum level may be valid for finite corrections (e.g., this is the case for the $4 - \epsilon$ renormalization group) or for sub-leading divergences at higher loops but not for the leading logarithms, including the one-loop divergences. Since the theory under

⁶Including the coefficient $(5/36)$, all coefficients agree with the previous calculation of [4].

consideration involves all degrees of freedom of the quantum metric, it is likely that the same situation holds in *any* theory of quantum gravity. Second, the enormous cancellation of the ρ -dependence (see Appendix C) in $\epsilon \rightarrow 0$ limit may be considered as a very strong test for the result (38) and [14]. Let us remark that Ref. [14] corrected the result of previous pioneering calculations [12, 4], and therefore an extra verification does not look unnecessary, especially in view of existing applications [42]. One has to notice that the difference between the results of [14] and [4] was only in R^2 -term and that the work [14] included an additional test (derivation using ζ -regularization in a space of constant curvature) for some special combination of this and C^2 -term. That test could be inefficient only for the error proportional to a combination $R_{\rho\lambda}^2 - \frac{1}{4}R^2$. Our calculation represents a very strong test of this possibility and one may be certain that the result (38) is a correct one. Third, our calculations have been organized in such a way that we could separate the result for the conformal case $\frac{1}{\xi} \rightarrow 0$ at every stage. Therefore, the coincidence of the final result with the one of [14] is an additional confirmation for our previous derivation of divergences in conformal quantum gravity [33], where we met a perfect agreement with the result of [34] and partial agreement with the one in [4]. Finally, the cancellation of a ρ -dependence in the $\epsilon \rightarrow 0$ limit provides a particular but rather strong test for the correctness of $\epsilon \neq 0$ coefficients presented in (33)-(37) and Appendix C. These coefficients will be applied in the next section for investigating the $4 - \epsilon$ renormalization group equations.

5 Renormalization group equations for higher derivative terms

Despite the ρ -dependence cancels out in the $\epsilon \rightarrow 0$ limit, the $4 - \epsilon$ renormalization group equations do not assume exactly the form of the known equations in four dimensions. The reason is that $4 - \epsilon$ renormalization group β -functions are sensitive to $\mathcal{O}(\epsilon)$ -corrections which depend on ρ . A standard derivation (see, e.g. [17]) yields the following renormalization group equations for the parameters of the higher derivative sector

$$\begin{aligned}\frac{d\rho}{dt} &= -\epsilon\rho + \beta_1\rho^2 \\ \frac{d\lambda}{dt} &= -\epsilon\lambda - 2\beta_2\lambda^2 \\ \frac{d\xi}{dt} &= -\epsilon\xi - \beta_3\xi^2,\end{aligned}\tag{39}$$

where μ is the renormalization parameter of dimensional regularization, $dt = d\mu/\mu$ and the β -functions are given by formulas (32). In the limit $\epsilon \rightarrow 0$, we meet usual four dimensional renormalization group equations, which are exactly the ones obtained earlier in [14]

$$\begin{aligned}(4\pi)^2 \frac{d\rho}{dt} &= -\frac{196}{45}\rho^2, & (4\pi)^2 \frac{d\lambda}{dt} &= -\frac{133}{10}\lambda^2, \\ (4\pi)^2 \frac{d\xi}{dt} &= -10\lambda^2 + 5\lambda\xi - \frac{5}{36}\xi^2.\end{aligned}\tag{40}$$

It is easy to see that the $n = 4 - \epsilon$ equations (39) are much more complicated than their 4-dimensional cousins (40). In order to see this, let us notice that the first two equations (40)

do not depend on parameter ξ . These two equations can be easily solved and their solutions replaced into the last equation, which can be also solved analytically [4, 14]. For convenience we present the known solution of all three equations and their analysis in Appendix D. However, the equations (39) do not admit any simple factorization and have to be solved simultaneously. Taking into account their complexity, there is no hope to achieve solution in the analytical form, and one has to apply numerical methods.

In fact, the situation with $4 - \epsilon$ renormalization group equations is even more complex, because one has to account for the arbitrariness coming from the choice of a gauge-fixing condition. We have already considered this aspect of theory in section 3. Taking the gauge-fixing arbitrariness (16) into account, we arrive at the complete form of $4 - \epsilon$ renormalization group equations for the three parameters

$$\frac{d\rho}{dt} = -\epsilon \rho + \epsilon \rho f(\alpha_i) + \rho^2 \beta_1, \quad (41)$$

$$\frac{d\lambda}{dt} = -\epsilon \lambda + \epsilon \lambda f(\alpha_i) - 2\lambda^2 \beta_2, \quad (42)$$

$$\frac{d\xi}{dt} = -\epsilon \xi + \epsilon \xi f(\alpha_i) - \xi^2 \beta_3, \quad (43)$$

where the β -functions (32) correspond to minimal gauge-fixing condition. The remarkable feature of the gauge-fixing dependent terms is that they are proportional to the same quantity $f(\alpha_i)$. Furthermore, the β -functions (32) are homogeneous functions on the couplings. Taking all that into account, our main strategy in the investigation of the system (41) – (43) will be the following. We construct two combinations θ and ω of the effective charges (λ, ρ, ξ) such that the renormalization group equations for these parameters are free from gauge-fixing ambiguity. The equations for the charges θ and ω will be explored with the main purpose of establishing the UV stable fixed points. After that we shall consider the equation for the remaining effective charge, with the invariant combinations θ and ω at the fixed point. In this way we shall learn the asymptotic UV behaviour corresponding to a given fixed point.

The most natural choice for independent effective charge is of course λ , for it defines the interaction between gravitons. Therefore we define the invariant charges as ratios between other parameters and λ

$$\theta = \frac{\lambda}{\rho}, \quad \omega = -\frac{3\lambda}{\xi}. \quad (44)$$

The coefficient in the second expression provides correspondence with the notations of [4, 14]. It is straightforward to see that the renormalization group equations for θ and ω are independent on function $f(\alpha_i)$ and have the following universal form:

$$\frac{d\theta}{d\tau} = -2\theta \beta_2 - \beta_1, \quad \frac{d\omega}{d\tau} = -2\omega \beta_2 - 3\beta_3. \quad (45)$$

Here $\tau(t)$ is a new parameter defined by

$$d\tau = \frac{\lambda(t)}{(4\pi)^2} dt, \quad (46)$$

where $\lambda(t)$ is a solution of (42).

Equation (46) can be viewed as a most important relation of the whole $4 - \epsilon$ approach. We know that $t \rightarrow \infty$ corresponds to the high energy limit $\mu \rightarrow \infty$ in the $\overline{\text{MS}}$ renormalization scheme. However, due to the gauge-fixing dependence of the renormalization group equation for $\lambda(t)$, it is not clear whether this high energy limit corresponds to a certain limit in the new variable τ . Perhaps even more important aspect of the same problem is the possible arbitrariness in the asymptotic behaviour of the coupling λ . This arbitrariness may put under question the asymptotic freedom of the theory, spoiling the physical sense of the renormalization group. We shall consider the asymptotic behaviour of λ later on and for a while, when looking for the new fixed points for ω and θ , will identify the limit $\tau \rightarrow \infty$ as the UV one.

Fixed Point	θ	ω	Stability
1	0.33516	-5.38892	Saddle
2	4.61183	-1.60198	UV-Unstable
3	-4.31710	-1.47066	UV-Unstable
4	-4.44192	-0.15162	Saddle
5	4.80565	-0.06229	Saddle
6	0.33782	-0.00283	UV-Stable
7	-3.94162	0.03123	UV-Unstable
8	-4.11072	0.07230	Saddle

Table 1. The list of the fixed points and their stability with respect to small perturbations for the case $\epsilon = 0.1$.

According to the outline described above, consider equations (45). For any given value of ϵ , this is a system of two nonlinear and rather complicated equations. There are no chances to obtain the general solution analytically, so the unique visible possibility is to fix the value of ϵ , according to the standard practice [27]. We shall try different values of ϵ , starting from $\epsilon = 0.1$. In this case the equations for the fixed points

$$2\theta\beta_2 + \beta_1 = 0, \quad 2\omega\beta_2 + 3\beta_3 = 0, \quad (47)$$

have eight distinct solutions collected in the Table 1. We investigated their stability under small perturbations of both charges $\theta_k \rightarrow \theta_k + \delta\theta_k$, $\omega_k \rightarrow \omega_k + \delta\omega_k$, where the values θ_k , ω_k , $k = 1, 2, \dots, 8$ correspond to different fixed points. It turns out that for $\epsilon = 0.1$ there are all kinds of the fixed points: UV stable, UV-unstable in all directions and saddle points.

In order to compare with the standard $\epsilon = 0$ case (see Appendix D), we present the corresponding fixed points in the Table 2. In this case there are only two fixed points, what means that six new fixed points have emerged due to the procedure $\epsilon = 0.1$. The renormalization group trajectories which result from numerical investigation are shown at Figure 1 for the simplest $\epsilon = 0$ case. Each arrow indicates the slope of integral curves, $d\theta/d\omega$ at a given point of the phase diagram.

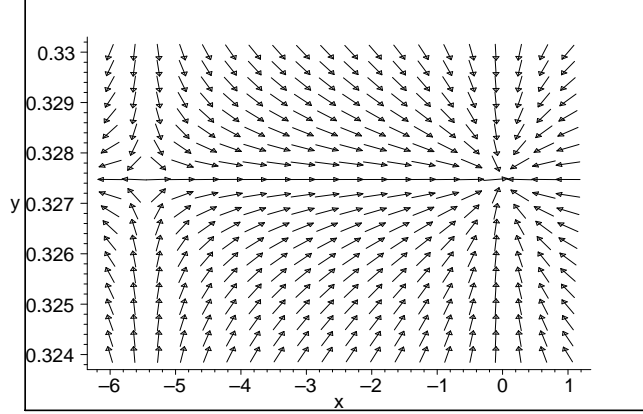


Figure 1: Diagram for $\epsilon = 0$ (four dimensions). Here y stands for θ and x for ω .

Fixed Point	θ	ω	Stability
1	0.32749	-0.02286	UV-Stable
2	0.32749	-5.4671	Saddle

Table 2. Fixed points for θ and ω and their stability for the simple case $\epsilon = 0$. The equations admit also complete analytical investigation presented in Appendix D.

For $\epsilon = 0.1$, the renormalization group flow is illustrated at Figure 2. Notice that the points 1 and 6 are shown in detail in the right diagram, which is very similar to the diagram at Figure 1. Indeed the trajectories in a limited region containing points 1 and 6 are practically the same trajectories carried out for $\epsilon = 0$. Thus, the procedure $\epsilon = 0.1$ keeps the dynamics unchanged in some limited region of the plane⁷, producing just a small displacement of the fixed points (in this sense, points 1 and 6 are direct descendants from the standard 4D-fixed points). However, in large scale, dynamics is totally modified by the appearance of six extra fixed points.

The situation for other small positive values of ϵ is qualitatively similar to the one in the $\epsilon = 0.1$ case. For example, for $\epsilon = 0.01$ we meet 10 fixed points, but only one of them is UV-stable. We will not present the details of this case here.

The results for the negative ϵ are quite different. The result of numerical integration for the cases $\epsilon = -0.1$ and $\epsilon = -0.01$ are shown at Figure 3. The general structure is similar to the solution in four dimensions, and no extra fixed point emerges by consequence of $\epsilon \neq 0$, only small displacements of these points take place.

Now we are in a position to explore the issue of asymptotic freedom in $4 - \epsilon$ framework. For this end we need to investigate the behavior of the effective charge λ in the UV limit $t \rightarrow \infty$. Also, this defines the physical sense of parameter $\tau(t)$ and its possible relation to the change of

⁷The meaning of this statement is somewhat restrictive: of course one has to consider only trajectories confined in that region.

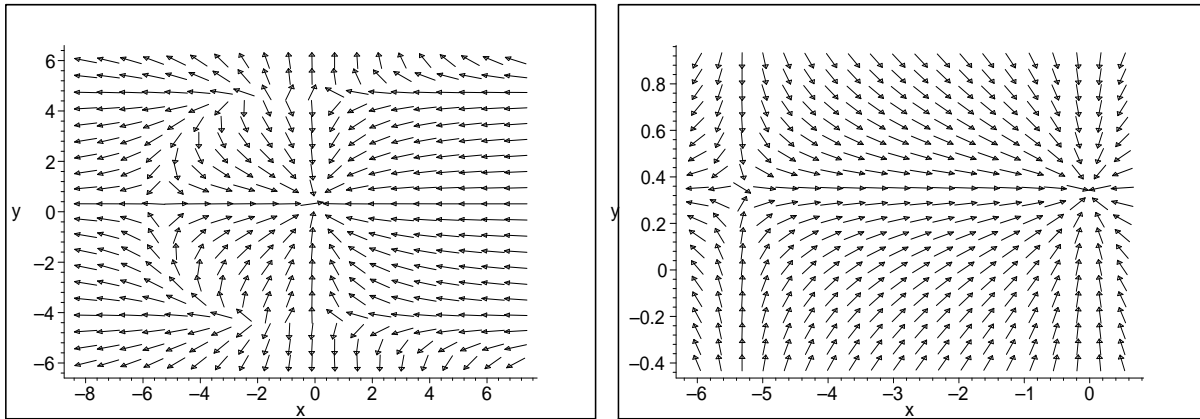


Figure 2: Numerical integration for $\epsilon = 0.1$ in the whole plane (ω, θ) in the left diagram. The right diagram shows a more detailed description of the trajectories between point 1 (saddle) and point 6, which is UV-stable.

a physical energy scale. Hence, this consideration should clarify the physical sense of equations (45) and their main characteristics, that are the corresponding fixed points.

Starting from the equation for $\lambda(t)$ in (39), one can find the analytical form of $\lambda(t)$ in the vicinity of a fixed point (ω_0, θ_0) . For this end, the expression β_2 must be rewritten in terms of ω and θ , which should be further replaced by ω_0 and θ_0 . After performing this, independent on the values of ω_0 and θ_0 we arrive at the equation

$$\frac{d\lambda}{dt} = a\lambda - b^2\lambda^2, \quad (48)$$

where

$$a = -\epsilon + \epsilon f(\alpha_i) \quad \text{and} \quad b^2 = 2\beta_2(\omega_0, \theta_0). \quad (49)$$

Let us remark that the parameter b in the last equation depends only on the values (ω_0, θ_0) , while the parameter a depends also on the choice of a gauge-fixing condition and therefore can be made arbitrary.

The solution of (48) is straightforward

$$\lambda(t) = \frac{a\lambda_0 e^{at}}{b\lambda_0 (e^{at} - 1) + a}, \quad \lambda_0 = \lambda(0). \quad (50)$$

Starting from this relation, we integrate (46) and arrive at the explicit form of τ

$$\tau(t) = \frac{1}{b} \ln [a + b\lambda_0 (e^{at} - 1)] + C, \quad (51)$$

where C is irrelevant integration constant.

It is easy to see that, formally, the asymptotic freedom in the theory depends on the sign of the quantity a in equation (49). In case $a > 0$ we have $\lambda(t) \rightarrow 0$ in the UV $t \rightarrow \infty$, and also

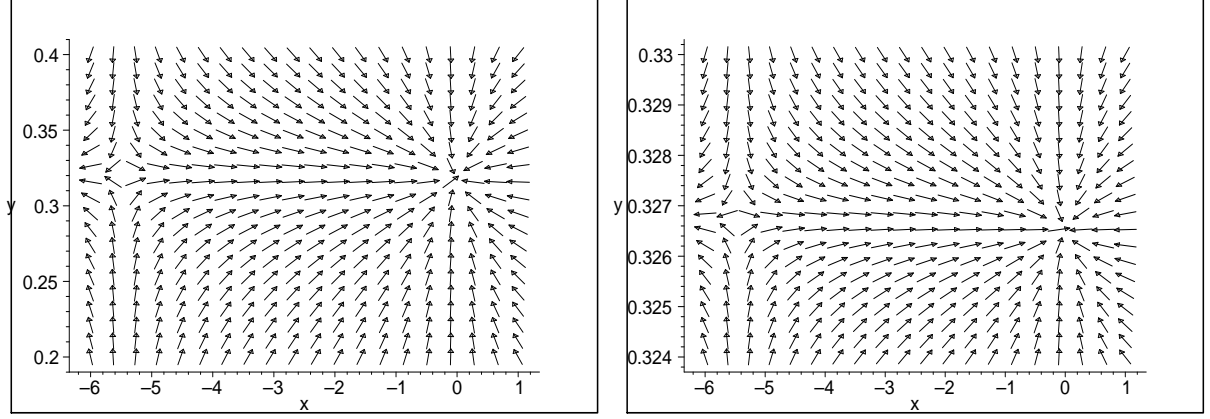


Figure 3: Numerical integration for $\epsilon = -0.1$ (left) and $\epsilon = -0.01$ (right), where the two fixed points in both cases are easily recognizable.

$\tau \rightarrow \infty$, so everything is consistent with the $\epsilon = 0$ case. On the opposite, for $a \leq 0$ we meet the asymptotic behaviour $\lambda(t) \rightarrow \lambda_0 \neq 0$ in the UV limit $t \rightarrow \infty$, also $\tau \rightarrow \tau_0 \neq \infty$.

It is important that the explicit form of the function $f(\alpha_i)$ derived in [40]

$$f(\alpha_i = \alpha, \gamma) = \frac{3}{2} \left\{ \frac{\lambda}{2} \ln \left[\frac{9\gamma\lambda^{5/2}}{2\alpha^{3/2}(\alpha - 6\lambda\gamma)} \right] + \alpha - \lambda \right\} \quad (52)$$

does not produce any restrictions for the value of $f(\alpha_i)$, which can be modified arbitrarily by choosing one or another gauge-fixing condition. According to equation (49), this means that one can also change the sign of the quantity a , and thus the asymptotic freedom in the $4 - \epsilon$ theory is not a physical phenomenon, but an artificial occurrence depending on the choice of the gauge-fixing condition.

Is it true that $4 - \epsilon$ renormalization group does not have physical sense for the case of higher derivative quantum gravity? In fact, this is a problem of interpretation. Let us remind that the original $4 - \epsilon$ approach [27] treats ϵ as a small parameter of the non-perturbative expansion. If we take this position and consider all terms proportional to ϵ as small *by definition*, then the asymptotic behavior for λ is close to $n = 4$ renormalization group equation (79). In this case the status of the new fixed points becomes clear and we conclude that the stable fixed points in all cases are similar to the one in $n = 4$ renormalization group. The same concerns also the renormalization group flows in the vicinity of the stable fixed points. At the same time, the existence of the numerous new unstable and saddle fixed points for $\epsilon > 0$ indicates the possibility of a rich non-perturbative structure of the theory.

6 Renormalization Group in the Einstein-Hilbert sector

Consider the renormalization group equations in the Einstein-Hilbert sector of the theory, including the cosmological constant. The renormalization group equations for the parameters in the low energy sector of theory are defined by the expressions for divergences (31), but must also account for the gauge-fixing dependence (16). The form of these equations is as follows:

$$\frac{d\kappa^2}{dt} = (n-4)\kappa^2 + \kappa^2\beta_4 + \frac{n-2}{2}\kappa^2 f(\alpha_i) \quad (53)$$

for the Newtonian constant and

$$\frac{d\Lambda}{dt} = \Lambda\beta_4 + \frac{1}{2}\Lambda\beta_5 + \frac{\beta_6}{2\kappa^2} - \Lambda f(\alpha_i) \quad (54)$$

for the cosmological constant.

The presence of the function $f(\alpha_i)$ in both equations indicates a strong gauge-fixing dependence. Indeed, there is nothing new in this dependence, as already known from [4] (see also [40]). The standard solution of this problem is to look for the gauge-fixing invariant combination of the two β -functions. It is assumed that the corresponding effective charge is an essential coupling constant [5]. Indeed, in the $n = 4$ case such combination is the dimensionless product $\kappa^2\Lambda$ of the two effective charges κ and Λ . The question is whether the essential effective charge can be found for the case of $4 - \epsilon$ renormalization group equation.

By direct computation one can easily obtain the unique combination

$$\gamma = \kappa^2\Lambda^N, \quad N = \frac{n-2}{2}$$

of κ and Λ which is explicitly independent on gauge-fixing parameters. As one could expect, this is exactly the combination of κ and Λ which is dimensionless for any value of n . Of course, for $n = 4$ we come back to the standard combination of parameters $\gamma(4) = \kappa^2\Lambda$.

Using (53) and (54) we arrive at the renormalization group equation for γ

$$\frac{d\gamma}{dt} = \gamma \left[n - 4 + \frac{1}{2} \left(2(N+1)\beta_4 + N\beta_5 + \frac{N}{\kappa^2\Lambda}\beta_6 \right) \right]. \quad (55)$$

The last equation looks like the one for the essential coupling, but in fact there is still a difficulty in its interpretation. The problem is that the expressions $\beta_{4,5,6}$ can be written in terms of essential couplings, ω and θ , but there will be some extra factors of $\lambda(t)$ in (55) such that it can not be absorbed in a redefinition of dt . Besides, the gauge dependent quantity $\kappa^2\Lambda$ (which is dimensionless only in four dimensions) also appears in the coefficient of β_6 . The consequence is that, without imposing the smallness of ϵ , the $(4 - \epsilon)$ -renormalization group for $\kappa^2\Lambda^N$ is gauge dependent, exactly as in the case of the higher derivative couplings. On the other hand, if we remember that all our consideration is called to model the non-perturbative ϵ -expansion and define ϵ as a small parameter, the renormalization group flow near the UV-stable fixed point ω_0, θ_0 will not have qualitative difference with the $\epsilon = 0$ case investigated in [4, 14]. The main feature of the UV asymptotic behaviour of γ is strong dependence on the initial data for the couplings ξ and γ .

The most interesting aspect of the renormalization group equation for the cosmological constant would be to study the low-energy limit. This investigation could provide better understanding of the possible role of quantum gravity for the cosmological constant problem [46, 37], including the remnant quantum effects of the particles with the mass of the Planck order of magnitude [47] and also for establishing the assumed universality property of quantum gravity [36]. However, the framework presented above is not appropriate for this purpose, because in the low-energy region we expect to meet a decoupling of the massive (ghost) mode (4) and the renormalization group equation should essentially modify. In order to observe decoupling one has to perform calculations and find β -functions in the physical mass-dependent renormalization scheme. Similar program has been realized recently for the massive matter fields on curved background [48]. The practical calculations in higher derivative quantum gravity, despite they are looking rather difficult from the technical point of view, represent a serious challenge for the future work.

7 Conclusions

We performed a complicated calculation of the counterterms in the fourth derivative quantum gravity. It is the first time that the quantum effects of the Gauss-Bonnet term have been taken into account. In the $n = 4 - \epsilon$ case the effects of this term are shown to be non-trivial, in accordance with the prediction by Capper and Kimber [32] concerning an important role of the topological term in quantum gravity. At the same time, for $n = 4$ the effects of the topological term cancel, in this limit we recover the known result [4, 14]. This coincidence represents very efficient verification of previous derivations of the $n = 4$ renormalization group equations.

The renormalization group equations for the parameters of theory in the $n = 4 - \epsilon$ are essentially more complicated than in the $n = 4$ case, in particular, the equations for different couplings do not separate. Moreover these equations manifest much stronger gauge-fixing dependence than in the standard $n = 4$ case. We separated the universal corner of theory, which is characterized by a number of new fixed points for $\epsilon > 0$. These fixed points may be related to the rich non-perturbative structure of the theory in the ϵ -expansion. Out of the small- ϵ approximation the equation for the most important coupling λ (the parameter of the loop expansion of the theory), manifests a strong gauge-fixing dependence which does not take place for the usual $n = 4$ renormalization group. From a purely formal point of view, the physical interpretation of the $n = 4 - \epsilon$ renormalization group looks unclear in this case. However the situation changes if we treat ϵ as a small parameter. Then the theory remains asymptotically free for $\epsilon \neq 0$ and moreover, as we already mentioned, has a number of new fixed points for $\epsilon > 0$.

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Appendix A

In this Appendix we collect bulky elements of the operator (26). The expression for $\hat{U} = (\hat{U})_{\mu\nu, \alpha\beta}$ has the following form:

$$\begin{aligned}
(\hat{U})_{\mu\nu, \alpha\beta} = & \frac{4}{y+4x} \left\{ \frac{3x}{2} g_{\nu\beta} R_{\mu\rho\lambda\sigma} R_{\alpha}^{\rho\lambda\sigma} + \frac{5x+y}{2} R_{\alpha\mu}^{\lambda\rho} R_{\lambda\nu\beta\rho} + \frac{x-y}{2} R_{\alpha\mu}^{\rho\lambda} R_{\nu\beta\rho\lambda} \right. \\
& + \frac{y-5x}{4} \left(R_{\mu\sigma} R_{\alpha\nu\beta}^{\sigma} + R_{\alpha\sigma} R_{\mu\beta\nu}^{\sigma} \right) + \frac{y+2x}{2} R_{\mu\alpha} R_{\nu\beta} - \frac{x}{2} g_{\alpha\beta} R_{\mu\theta\sigma\tau} R_{\nu}^{\theta\sigma\tau} \\
& - \frac{1}{4} S^2 (\delta_{\mu\nu, \alpha\beta} - \Omega_1 g_{\mu\nu} g_{\alpha\beta}) + \left(\frac{3zR}{2} - \frac{3}{4\kappa^2} \right) g_{\nu\beta} R_{\alpha\mu} + \frac{3y}{2} g_{\nu\beta} R_{\mu}^{\lambda} R_{\alpha\lambda} \\
& + \frac{x+3y}{4} g_{\mu\nu} R^{\sigma\tau} R_{\alpha\sigma\beta\tau} + \left(\frac{zR}{2} - \frac{1}{4\kappa^2} \right) R_{\mu\alpha\nu\beta} + \frac{3x+y}{2} R_{\mu}^{\sigma} R_{\nu}^{\tau} R_{\tau\alpha\sigma\beta} \\
& - \frac{y}{2} g_{\alpha\beta} R_{\mu}^{\tau} R_{\nu\tau} - x \Omega_1 g_{\mu\nu} R_{\alpha\theta\sigma\tau} R_{\beta}^{\theta\sigma\tau} - \Omega_2 g_{\mu\nu} R_{\alpha}^{\tau} R_{\beta\tau} + z R_{\mu\nu} R_{\alpha\beta} \\
& + \left(\frac{1}{4\kappa^2} - \frac{zR}{2} \right) g_{\alpha\beta} R_{\mu\nu} + \left(\frac{1}{\kappa^2} \Omega_3 - z R \Omega_1 \right) g_{\mu\nu} R_{\alpha\beta} \\
& \left. - \frac{1}{4\kappa^2} \Omega (4\Lambda - R) g_{\mu\nu} g_{\alpha\beta} \right\}, \tag{56}
\end{aligned}$$

where

$$S^2 = x R_{\rho\lambda\sigma\tau}^2 + y R_{\rho\sigma}^2 + z R^2 - \frac{1}{\kappa^2} (R - 2\Lambda), \tag{57}$$

and the coefficients $\Omega_{1,2,3}$ are given by

$$\begin{aligned}
\Omega_1 &= \frac{2x+3y+10z}{\Sigma}, & \Omega_2 &= \frac{7xy+9y^2-4xz+28yz}{4\Sigma}, \\
\Omega_3 &= \frac{x+y+3z}{\Sigma}, & \text{with } \Sigma &= 4x-4z+n(y+4z). \tag{58}
\end{aligned}$$

For the matrix $\hat{V}^{\rho\lambda} = (\hat{V}^{\rho\lambda})_{\mu\nu, \alpha\beta}$ we have the expression

$$\hat{V}^{\rho\lambda} = \frac{4}{y+4x} \sum_{i=1}^{20} b_i \mathbf{k}_i, \tag{59}$$

where the following condensed notations have been used:

$$\begin{aligned}
\mathbf{k}_1 &= g_{\nu\beta} g^{\rho\lambda} R_{\mu\alpha}, & \mathbf{k}_2 &= \delta_{\mu\nu, \alpha\beta} g^{\rho\lambda}, & \mathbf{k}_3 &= g^{\rho\lambda} R_{\mu\alpha\nu\beta}, \\
\mathbf{k}_4 &= \delta_{\nu\beta, \rho\lambda} R_{\mu\alpha}, & \mathbf{k}_5 &= \delta_{\nu\beta, \rho\lambda} g_{\mu\alpha}, & \mathbf{k}_6 &= \delta_{\mu\nu, \alpha\beta} R^{\rho\lambda}, \\
\mathbf{k}_7 &= \frac{1}{2} (\delta_{\nu}^{(\rho} R^{\lambda)_{\alpha\beta\mu}} + \delta_{\beta}^{(\rho} R^{\lambda)_{\mu\nu\alpha}}), & \mathbf{k}_8 &= g_{\nu\beta} \delta_{(\mu}^{(\rho} R^{\lambda)_{\alpha}}), \\
\mathbf{k}_9 &= g_{\nu\beta} R_{(\alpha}^{\rho\lambda}{}_{\mu)}, & \mathbf{k}_{10} &= \frac{1}{2} (\delta_{\alpha\beta, \rho\lambda} R_{\mu\nu} + \delta_{\mu\nu, \rho\lambda} R_{\alpha\beta}), \\
\mathbf{k}_{11} &= g_{\mu\nu} R_{\alpha}^{\rho\lambda}{}_{\beta}, & \mathbf{k}_{12} &= g_{\alpha\beta} R_{\mu}^{\rho\lambda}{}_{\nu}, & \mathbf{k}_{13} &= g_{\mu\nu} g^{\rho\lambda} R_{\alpha\beta}, \\
\mathbf{k}_{14} &= g_{\alpha\beta} g^{\rho\lambda} R_{\mu\nu}, & \mathbf{k}_{15} &= g_{\mu\nu} \delta_{\alpha}^{\lambda} R_{\beta}^{\rho}, & \mathbf{k}_{16} &= g_{\alpha\beta} \delta_{\mu}^{\lambda} R_{\nu}^{\rho}, \\
\mathbf{k}_{17} &= g_{\mu\nu} \delta_{\alpha\beta, \rho\lambda}, & \mathbf{k}_{18} &= g_{\alpha\beta} \delta_{\mu\nu, \rho\lambda}, & \mathbf{k}_{19} &= g_{\mu\nu} g_{\alpha\beta} g^{\rho\lambda}, \\
\mathbf{k}_{20} &= g_{\mu\nu} g_{\alpha\beta} R^{\rho\lambda} \tag{60}
\end{aligned}$$

and

$$\begin{aligned}
b_1 &= -2x, & b_2 &= \frac{zR}{2} - \frac{1}{4\kappa^2}, & b_3 &= 3x + y, & b_4 &= 2x \\
b_5 &= \frac{1}{2\kappa^2} - zR, & b_6 &= x + \frac{y}{2}, & b_7 &= -4x, & b_8 &= -4x - 2y, \\
b_9 &= -2x, & b_{10} &= -2z, & b_{11} &= 4x\Omega_3, & b_{12} &= x, \\
b_{13} &= -y\Omega_3, & b_{14} &= z, & b_{15} &= 2y\Omega_3, & b_{16} &= \frac{y}{2}, \\
b_{17} &= 2zR\Omega_3 - \frac{\Omega_1 - 2\Omega}{2\kappa^2}, & b_{18} &= \frac{zR}{2} - \frac{1}{4\kappa^2}, & b_{19} &= -b_{17}, \\
b_{20} &= -y\Omega_3.
\end{aligned} \tag{61}$$

The last expressions have a non-symmetric form, and the rule of symmetrization is the one described in the section 4.

Appendix B. Particular results for necessary traces

In this Appendix we collect some intermediate formulas, which are necessary for the computation of the one-loop contributions. The calculations has been verified by using the package MathTensor [45] driven by the software Mathematica [44].

The trace of the operator \hat{U} is given by

$$\text{tr } \hat{U} = \delta^{\mu\nu, \alpha\beta} U_{\mu\nu, \alpha\beta} = A_1 R_{\mu\nu\alpha\beta}^2 + A_2 R_{\mu\nu}^2 + A_3 R^2 + A_4 \frac{R}{\kappa^2} + A_5 \frac{\Lambda}{\kappa^2}, \tag{62}$$

where

$$A_k = \frac{p_{k1} + p_{k2}n + p_{k3}n^2 + p_{k4}n^3}{2(4x + y)(4x - 4z + ny + 4nz)}, \quad k = 1, 2, 3, 4, 5$$

and

$$\begin{aligned}
p_{11} &= 8(8x^2 - 20xz - 3yz), & p_{12} &= 2(12x^2 + 13xy + 3y^2 + 40xz + 12yz), \\
p_{13} &= -x(4x - 5y - 24z), & p_{14} &= -x(y + 4z), \\
p_{21} &= 8(12x^2 - 3y^2 - 8xz - 12yz - 4z^2), & p_{22} &= 2(24xy + 5y^2 + 48xz + 12yz + 16z^2), \\
p_{23} &= -y(4x - 5y - 24z), & p_{24} &= -y(y + 4z), \\
p_{31} &= 8(2x^2 + xy - 4xz - 4yz - 10z^2), & p_{32} &= 2(2xy + y^2 + 20xz + 7yz), \\
p_{33} &= -z(4x - 5y - 24z), & p_{34} &= -z(y + 4z), \\
p_{41} &= 8(x + y + 3z), & p_{42} &= -4(y + z + 3x), & p_{43} &= 4x - 2y - 12z, & p_{44} &= y + 4z, \\
p_{51} &= 0, & p_{52} &= 4(y + 4z), & p_{53} &= -2(4x + y), & p_{54} &= -2(y + 4z).
\end{aligned}$$

The expression for $\text{tr } (\hat{V}_\gamma^\gamma R)$ can be presented as

$$\text{tr } (\hat{V}_\gamma^\gamma R) = B_1 R^2 + \frac{B_2}{\kappa^2} R, \tag{63}$$

where

$$B_i = \frac{l_{i1} + l_{i2}n + l_{i3}n^2 + l_{i4}n^3 + l_{i5}n^4}{2(4x + y)(4x - 4z + ny + 4nz)}, \quad i = 1, 2$$

and

$$\begin{aligned}
l_{11} &= -16(4x - y - 2z)(2x + y + 2z), \\
l_{12} &= -4(16x^2 + 20xy + 5y^2 - 4xz + 6yz + 12z^2), \\
l_{13} &= -16x^2 - 8xy - 10y^2 - 72xz - 56yz - 8z^2, \\
l_{14} &= -4xy + 2y^2 - 8xz + 6yz - 16z^2, & l_{15} &= 2z(y + 4z); \\
l_{21} &= 0, & l_{22} &= -4(2x + 2z + y), & l_{23} &= 4(3x + z + y), \\
l_{24} &= y - 4x + 8z, & l_{25} &= -y - 4z;
\end{aligned}$$

The computation of $\text{tr}(\hat{V}^{\rho\lambda}R_{\rho\lambda})$ yields

$$\text{tr}(\hat{V}^{\rho\lambda}R_{\rho\lambda}) = C_6 R_{\mu\nu}^2 + C_7 R^2 + C_3 \frac{R}{\kappa^2}, \quad (64)$$

where

$$\begin{aligned}
C_i &= \frac{p_{i1} + p_{i2}n + p_{i3}n^2 + p_{i4}n^3}{(4x + y)(4x - 4z + n(y + 4z))}, \quad i = 6, 7, \\
C_3 &= \frac{-(n^2 - 3n + 2)(4x + ny + 2y + 4z + 4nz)}{2(4x + y)(4x - 4z + n(y + 4z))},
\end{aligned}$$

with

$$\begin{aligned}
p_{61} &= -8(4x - y - 2z)(2x + y + 2z), \\
p_{62} &= -2(4x^2 + 14xy + 3y^2 + 20xz + 8yz + 16z^2), \\
p_{63} &= 8x^2 + 2xy - 3y^2 - 16xz - 16yz, & p_{64} &= (2x + y)(y + 4z), \\
p_{71} &= -4(6x^2 + 3xy + y^2 - 12xz - yz - 2z^2), \\
p_{72} &= -2(8x^2 + 3xy + y^2 + 10xz + 6yz + 2z^2), \\
p_{73} &= -4xy - 12xz - yz - 8z^2 & p_{74} &= z(y + 4z).
\end{aligned}$$

The results for the traces $\frac{1}{48}\text{tr}(\hat{V}_\lambda^\lambda \hat{V}_\rho^\rho) + \frac{1}{24}\text{tr}(\hat{V}_{\lambda\rho} \hat{V}^{\lambda\rho})$ can be written as

$$\frac{1}{48}\text{tr}(\hat{V}_\lambda^\lambda \hat{V}_\rho^\rho) + \frac{1}{24}\text{tr}(\hat{V}_{\lambda\rho} \hat{V}^{\lambda\rho}) = D_1 R_{\mu\nu\alpha\beta}^2 + D_2 R_{\mu\nu}^2 + D_3 R^2 + D_4 \frac{R}{\kappa^2} + D_5 \frac{1}{\kappa^4}, \quad (65)$$

where

$$\begin{aligned}
D_1 &= \frac{q_{11} + q_{12}n + q_{13}n^2 + q_{14}n^3}{96(4x + y)^2(4x - 4z + n(y + 4z))}, \\
D_i &= \frac{q_{i1} + q_{i2}n + q_{i3}n^2 + q_{i4}n^3 + q_{i5}n^4 + q_{i6}n^5 + q_{i7}n^6}{96(4x + y)^2(4x - 4z + n(y + 4z))^2}, \quad i = 2, 3, 4, 5,
\end{aligned} \quad (66)$$

with all non-zero q_{ij} given by

$$\begin{aligned}
q_{11} &= 1536x(2x^2 - 6xz - yz), \\
q_{12} &= 192(26x^3 + 18x^2y + 3xy^2 + 6x^2z - 2xyz - y^2z), \\
q_{13} &= 48(18x^3 + 36x^2y + 12xy^2 + y^3 + 78x^2z + 28xyz + 2y^2z),
\end{aligned}$$

$$\begin{aligned}
q_{14} &= 24 (3x + y)^2 (y + 4z) , \\
q_{21} &= -256 (8x^4 + 2x^2y^2 - 4xy^3 - y^4 + 64x^3z + 112x^2yz + 20xy^2z \\
&\quad - 160x^2z^2 - 80xyz^2 - 10y^2z^2 + 32xz^3 - 8z^4) , \\
q_{22} &= 128 (120x^4 + 84x^3y + 27x^2y^2 + 6xy^3 + y^4 - 224x^3z - 148x^2yz - 54xy^2z \\
&\quad - 6y^3z - 136x^2z^2 - 148xyz^2 - 33y^2z^2 + 80xz^3 - 12yz^3 - 32z^4) , \\
q_{23} &= 16 (448x^4 + 704x^3y + 360x^2y^2 + 64xy^3 + 15y^4 + 1024x^3z + 576x^2yz \\
&\quad + 8xy^2z + 84y^3z - 1344x^2z^2 - 816xyz^2 + 116y^2z^2 + 64xz^3 + 144yz^3 + 128z^4) , \\
q_{24} &= 8 (64x^4 + 448x^3y + 344x^2y^2 + 104xy^3 + 5y^4 + 1664x^3z + 1408x^2yz \\
&\quad + 560xy^2z + 28y^3z + 320x^2z^2 + 512xyz^2 + 8y^2z^2 - 512xz^3 - 128yz^3) , \\
q_{25} &= 8 (y + 4z) (32x^3 + 56x^2y + 28xy^2 + 3y^3 + 192x^2z + 112xyz \\
&\quad + 32xz^2 + 12y^2z + 8yz^2) , \quad q_{26} = 32x^2(y + 4z)^2 , \\
q_{31} &= -128 (56x^4 + 32x^3y + 18x^2y^2 + 6xy^3 + y^4 - 144x^3z - 48x^2yz \\
&\quad - 2y^3z - 24xy^2z + 240x^2z^2 - 6y^2z^2 - 80xz^3 - 16yz^3 - 8z^4) , \\
q_{32} &= -64 (68x^4 + 96x^3y + 36x^2y^2 + 13xy^3 + 3y^4 + 72x^3z - 48x^2yz + 10xy^2z \\
&\quad + 13y^3z - 216x^2z^2 - 16xyz^2 + 26y^2z^2 + 168xz^3 + 64yz^3 + 36z^4) , \\
q_{33} &= -8 (32x^4 + 304x^3y + 204x^2y^2 + 44xy^3 + y^4 + 1088x^3z + 784x^2yz + 280xy^2z \\
&\quad + 4y^3z - 1088x^2z^2 - 80xyz^2 - 144y^2z^2 + 192xz^3 - 336yz^3 - 160z^4) , \\
q_{34} &= -4 (32x^3y + 100x^2y^2 + 32xy^3 + 7y^4 + 192x^3z + 640x^2yz + 208xy^2z \\
&\quad + 1392x^2z^2 + 68y^3z + 448xyz^2 + 216y^2z^2 - 544xz^3 + 64yz^3 - 80z^4) , \\
q_{35} &= 4 (-4x^2y^2 - 4xy^3 + y^4 - 64x^2yz - 24xy^2z + 6y^3z - 144x^2z^2 \\
&\quad - 18y^2z^2 - 8xyz^2 - 136yz^3 - 80z^4) , \\
q_{36} &= -4z (y + 4z) (4xy - 2y^2 + 8xz - 9yz + 4z^2) , \quad q_{37} = 4z^2(y + 4z)^2 , \\
q_{41} &= -128 (8x^3 + 4x^2y + 6xy^2 + y^3 - 80x^2z - 8xyz + 40xz^2 + 4yz^2) , \\
q_{42} &= -32 (16x^3 - 8xy^2 - y^3 + 120x^2z - 2y^2z - 160xz^2 - 32yz^2 - 8z^3) , \\
q_{43} &= 16 (16x^3 - 8x^2y + 16xy^2 + y^3 - 176x^2z + 24xyz - 14y^2z \\
&\quad + 80xz^2 - 48yz^2 - 16z^3) , \\
q_{44} &= 8 (16x^3 - 12xy^2 + y^3 + 120x^2z - 16xyz + 16y^2z - 160xz^2 - 40z^3) , \\
q_{45} &= 4 (16x^2y - 4xy^2 + y^3 + 16x^2z - 40xyz + 18y^2z + 88yz^2 + 80z^3) , \\
q_{46} &= -4 (y + 4z) (-2xy + y^2 + 5yz - 4z^2) , \quad q_{47} = -4z(y + 4z)^2 , \\
q_{51} &= 0 , \quad q_{52} = 16 (6x + y - 2z) (2x + y + 2z) , \\
q_{53} &= -8 (24x^2 + 12xy + y^2 - 4yz - 8z^2) , \\
q_{54} &= -8 (6x^2 + 8xy + y^2 + 20xz - 10z^2) , \\
q_{55} &= 2 (24x^2 + 12xy - y^2 - 20yz - 40z^2) , \\
q_{56} &= (8x + y - 4z) (y + 4z) , \quad q_{57} = (y + 4z)^2 .
\end{aligned}$$

Finally, let us present details of the contributions from Faddeev-Popov ghosts and weight operator (sometimes called third ghost). The action of the ghosts has the form

$$S_{gh} = \int d^4x \sqrt{g} \bar{C}^\mu (\mathcal{H}_{gh})_\mu^\nu C_\nu, \quad (67)$$

where the operator $\hat{\mathcal{H}}_{gh}$ is given by the expression

$$\hat{\mathcal{H}}_{gh} = (\mathcal{H}_{gh})_\beta^\alpha = \frac{\delta \chi^\alpha}{\delta h^{\mu\nu}} \mathcal{D}^{\mu\nu}{}_{,\beta}.$$

Here $\mathcal{D}^{\mu\nu}{}_{,\beta}$ is generator of the diffeomorphism transformations for the metric $\delta g^{\mu\nu} = \mathcal{D}^{\mu\nu}{}_{,\beta} \delta x^\beta$

$$\mathcal{D}_{\mu\nu}{}_{,\beta} = -g_{\mu\beta} \nabla_\nu - g_{\nu\beta} \nabla_\mu.$$

By direct computation one derives, assuming $p_1 = p_2 = 0$ in (8) and keeping other gauge-fixing parameters arbitrary (10)

$$\hat{\mathcal{H}}_{gh} = (\mathcal{H}_{gh})_\mu^\nu = -\delta_\mu^\nu \square - \nabla^\nu \nabla_\mu - 2\beta \nabla_\mu \nabla^\nu. \quad (68)$$

In order to evaluate the expression $-i \ln \text{Det} \hat{\mathcal{H}}_{gh}$, let us rewrite (68) in the form suitable for the generalized Schwinger-DeWitt method [38]

$$\hat{\mathcal{H}}_{gh} = -[\delta_\mu^\nu \square - \sigma \nabla_\mu \nabla^\nu + P_\mu^\nu], \quad (69)$$

where $\sigma = -(1 + 2\beta)$ and $P_{\mu\nu} = R_{\mu\nu}$. The n dimensional analog of the known algorithm [4, 38] for the non-minimal Abelian vector operator (69) has the form

$$\begin{aligned} -\frac{i}{2} \ln \text{Det} \hat{\mathcal{H}}_{gh}|_{\text{div}} &= \frac{\mu^{n-4}}{(4\pi)^2(n-4)} \int d^n x \sqrt{g} \left\{ \frac{n-15}{180} R_{\mu\nu\alpha\beta}^2 + \left(\frac{1}{24} \psi^2 + \frac{1}{12} \psi - \frac{n}{180} \right) R_{\mu\nu}^2 \right. \\ &+ \left(\frac{1}{48} \psi^2 + \frac{1}{12} \psi + \frac{n}{72} \right) R^2 + \left(\frac{1}{12} \psi^2 + \frac{1}{3} \psi \right) R_{\mu\nu} P^{\mu\nu} \\ &+ \left. \left(\frac{1}{24} \psi^2 + \frac{1}{4} \psi + \frac{1}{2} \right) P_{\mu\nu}^2 + \left(\frac{1}{24} \psi^2 + \frac{1}{12} \psi + \frac{1}{6} \right) R P + \frac{1}{48} \psi^2 P^2 \right\}, \quad (70) \end{aligned}$$

where $P = P_\alpha^\alpha$ and

$$\psi = \frac{\sigma}{1-\sigma} = -\frac{1+2\beta}{2+2\beta}.$$

Taking into account $P_{\mu\nu} = R_{\mu\nu}$, we find

$$\begin{aligned} -i \ln \text{Det} \hat{\mathcal{H}}_{gh}|_{\text{div}} &= \frac{\mu^{n-4}}{(4\pi)^2(n-4)} \int d^n x \sqrt{g} \left\{ \frac{n-15}{90} R_{\mu\nu\alpha\beta}^2 \right. \\ &+ \left. \left(\frac{1}{3} \psi^2 + \frac{4}{3} \psi + \frac{90-n}{90} \right) R_{\alpha\beta}^2 + \left(\frac{1}{6} \psi^2 + \frac{1}{3} \psi + \frac{n+12}{36} \right) R^2 \right\}. \quad (71) \end{aligned}$$

The contribution of the weight operator (8) is calculated with the choice (20). The divergent part of $\text{Tr} \ln \hat{Y}$ can be computed directly from (70). As usual [4, 38], in the case $P_{\mu\nu} = -R_{\mu\nu}$, the gauge dependence is canceled (this cancellation is related to the gauge invariance of the electromagnetic field, whose contribution is given by exactly the same operator):

$$-\frac{i}{2} \ln \text{Det} \hat{Y}|_{\text{div}} = \frac{\mu^{n-4}}{(4\pi)^2(n-4)} \int d^n x \sqrt{g} \left\{ \frac{n-15}{180} R_{\mu\nu\alpha\beta}^2 + \frac{90-n}{180} R_{\alpha\beta}^2 + \frac{n-12}{72} R^2 \right\}. \quad (72)$$

Appendix C. Coefficients of β -functions in $4 - \epsilon$ dimension

This Appendix contains the coefficients δ of the β -functions (32), most of them were not included into the main text. The total expressions for the β -functions is as follows (32)

$$\begin{aligned}
(4\pi)^2 \beta_1(\rho, \lambda, \xi, \epsilon) &= \delta_{1A}^{(0)} + \delta_{1B}^{(0)} \frac{\lambda}{\xi} - \delta_{1A}^{(1)} \frac{\xi}{\rho} - \delta_{1B}^{(1)} \frac{\lambda}{\rho} + \delta_{1A}^{(2)} \frac{\xi^2}{\rho^2} + \delta_{1B}^{(2)} \frac{\lambda \xi}{\rho^2} + \delta_{1C}^{(2)} \frac{\lambda^2}{\rho^2}, \\
(4\pi)^2 \beta_2(\rho, \lambda, \xi, \epsilon) &= \delta_{2A}^{(0)} \frac{\xi}{\lambda} + \delta_{2B}^{(0)} + \delta_{2C}^{(0)} \frac{\lambda}{\xi} - \delta_{2A}^{(1)} \frac{\xi}{\rho} - \delta_{2B}^{(1)} \frac{\lambda}{\rho} + \delta_{2A}^{(2)} \frac{\xi^2}{\rho^2} + \delta_{2B}^{(2)} \frac{\lambda \xi}{\rho^2} + \delta_{2C}^{(2)} \frac{\lambda^2}{\rho^2}, \\
(4\pi)^2 \beta_3(\rho, \lambda, \xi, \epsilon) &= \delta_{3A}^{(0)} + \delta_{3B}^{(0)} \frac{\lambda}{\xi} + \delta_{3C}^{(0)} \frac{\lambda^2}{\xi^2} - \delta_{3A}^{(1)} \frac{\xi}{\rho} - \delta_{3B}^{(1)} \frac{\lambda}{\rho} - \delta_{3C}^{(1)} \frac{\lambda^2}{\xi \rho} \\
&\quad + \delta_{3A}^{(2)} \frac{\xi^2}{\rho^2} + \delta_{3B}^{(2)} \frac{\lambda \xi}{\rho^2} + \delta_{3C}^{(2)} \frac{\lambda^2}{\rho^2}.
\end{aligned} \tag{73}$$

The coefficients $\delta_{1A,B,C}^{(i)}$ have the form

$$\begin{aligned}
\delta_{1A}^{(0)} &= \frac{37632 - 92704\epsilon + 81860\epsilon^2 - 31350\epsilon^3 + 5533\epsilon^4 - 446\epsilon^5 + 15\epsilon^6}{2880(\epsilon - 3)(\epsilon - 1)^2}, \\
\delta_{1B}^{(0)} &= -\frac{\epsilon(\epsilon - 6)(\epsilon - 2)^2}{48(\epsilon - 1)^2}, \\
\delta_{1A}^{(1)} &= -\frac{\epsilon(\epsilon + 4)}{12(\epsilon - 3)^2}, \\
\delta_{1B}^{(1)} &= \frac{\epsilon(\epsilon - 2)(92 - 176\epsilon + 57\epsilon^2 - 10\epsilon^3 + \epsilon^4)}{48(\epsilon - 3)(\epsilon - 1)^2}, \\
\delta_{1A}^{(2)} &= -\frac{\epsilon^2(\epsilon - 2)(\epsilon - 1)}{96(\epsilon - 3)^4}, \\
\delta_{1B}^{(2)} &= -\frac{\epsilon^2(\epsilon - 5)(\epsilon - 4)(\epsilon - 2)}{48(\epsilon - 3)^3}, \\
\delta_{1C}^{(2)} &= \frac{\epsilon^2(\epsilon - 5)(\epsilon - 2)^2(\epsilon^2 - 12\epsilon + 29)}{48(\epsilon - 3)^2(\epsilon - 1)^2}.
\end{aligned} \tag{74}$$

For the expressions $\delta_{2A,B,C}^{(i)}$, we have

$$\begin{aligned}
\delta_{2A}^{(0)} &= -\frac{\epsilon(\epsilon^2 + 4\epsilon - 4)}{32(\epsilon - 3)^2(\epsilon - 1)}, \\
\delta_{2B}^{(0)} &= -\frac{(\epsilon - 2)(-9576 + 21180\epsilon - 15410\epsilon^2 + 4410\epsilon^3 - 399\epsilon^4 + 5\epsilon^5)}{960(\epsilon - 3)(\epsilon - 1)^2}, \\
\delta_{2C}^{(0)} &= \frac{\epsilon(\epsilon - 6)(\epsilon - 2)^2}{48(\epsilon - 1)^2}, \\
\delta_{2A}^{(1)} &= \frac{\epsilon(\epsilon - 2)(2\epsilon + 1)}{24(\epsilon - 3)^2(\epsilon - 1)}, \\
\delta_{2B}^{(1)} &= -\frac{\epsilon(\epsilon - 7)(\epsilon - 2)^2(\epsilon^2 - 4\epsilon + 10)}{48(\epsilon - 3)(\epsilon - 1)^2},
\end{aligned}$$

$$\begin{aligned}
\delta_{2A}^{(2)} &= \frac{\epsilon^2 (\epsilon - 2) (\epsilon - 1)}{96 (\epsilon - 3)^4}, \\
\delta_{2B}^{(2)} &= \frac{\epsilon^2 (\epsilon - 2)^2 (\epsilon^2 - 8\epsilon + 19)}{48 (\epsilon - 3)^3 (\epsilon - 1)}, \\
\delta_{2C}^{(2)} &= -\frac{\epsilon (\epsilon - 2)^3 (-162 + 95\epsilon - 18\epsilon^2 + \epsilon^3)}{48 (\epsilon - 3)^2 (\epsilon - 1)^2}.
\end{aligned} \tag{75}$$

The coefficients $\delta_{3A,B,C}^{(i)}$ can be written as

$$\begin{aligned}
\delta_{3A}^{(0)} &= \frac{-1200 + 8808\epsilon - 12480\epsilon^2 + 5980\epsilon^3 - 1290\epsilon^4 + 127\epsilon^5 - 5\epsilon^6}{960 (\epsilon - 3)^2 (\epsilon - 1)}, \\
\delta_{3B}^{(0)} &= -\frac{(\epsilon - 2)^2 (180 + 28\epsilon - 16\epsilon^2 + \epsilon^3)}{48 (\epsilon - 3) (\epsilon - 1)}, \\
\delta_{3C}^{(0)} &= \frac{(\epsilon - 6) (\epsilon - 5) (\epsilon - 4) (\epsilon - 2)^3}{96 (\epsilon - 1)^2}, \\
\delta_{3A}^{(1)} &= \frac{\epsilon (\epsilon - 2) (\epsilon - 1) (\epsilon^2 - 20)}{96 (\epsilon - 3)^3}, \\
\delta_{3B}^{(1)} &= -\frac{\epsilon (\epsilon - 2)^2 (-10 + 24\epsilon - 9\epsilon^2 + \epsilon^3)}{24 (\epsilon - 3)^2 (\epsilon - 1)}, \\
\delta_{3C}^{(1)} &= \frac{\epsilon (\epsilon - 6) (\epsilon - 5) (\epsilon - 2)^3}{48 (\epsilon - 3) (\epsilon - 1)}, \\
\delta_{3A}^{(2)} &= \frac{\epsilon^2 (\epsilon - 5) (\epsilon - 2)^2 (\epsilon - 1)^2}{192 (\epsilon - 3)^5}, \\
\delta_{3B}^{(2)} &= \frac{\epsilon^2 (\epsilon - 5) (\epsilon - 2)^2 (\epsilon - 1)}{48 (\epsilon - 3)^4}, \\
\delta_{3C}^{(2)} &= \frac{\epsilon^2 (\epsilon - 5) (\epsilon - 2)^3 (\epsilon^2 - 8\epsilon + 19)}{96 (\epsilon - 3)^3 (\epsilon - 1)}.
\end{aligned} \tag{76}$$

The coefficients $\delta_{4A,B,C}^{(i)}$ are the following:

$$\begin{aligned}
\delta_{4A}^{(0)} &= -\frac{(\epsilon - 2) (72 - 40\epsilon - 4\epsilon^2 + \epsilon^3)}{192 (\epsilon - 3)^2}, \\
\delta_{4B}^{(0)} &= \frac{(\epsilon - 2)^2 (-156 + 144\epsilon - 27\epsilon^2 + \epsilon^3)}{96 (\epsilon - 3) (\epsilon - 1)}, \\
\delta_{4C}^{(0)} &= -\frac{(\epsilon - 6) (\epsilon - 5) (\epsilon - 4) (\epsilon - 2)^3}{96 (\epsilon - 1)^2}, \\
\delta_{4A}^{(1)} &= -\frac{\epsilon (\epsilon - 6) (\epsilon - 2)^2 (\epsilon - 1)}{192 (\epsilon - 3)^3}, \\
\delta_{4B}^{(1)} &= -\frac{\epsilon (\epsilon - 6) (\epsilon - 5) (\epsilon - 2)^3}{96 (\epsilon - 3) (\epsilon - 1)},
\end{aligned} \tag{77}$$

The remaining coefficients corresponding to $\delta_{5A,B,C}^{(i)}$ and $\delta_{6A,B,C}^{(i)}$ have the form

$$\delta_{5A}^{(0)} = -\frac{(\epsilon - 4) (\epsilon - 2)}{4 (\epsilon - 3)^2}, \quad \delta_{5B}^{(0)} = \frac{(\epsilon - 4) (\epsilon - 2) (\epsilon^2 - 8\epsilon + 14)}{2 (\epsilon - 3) (\epsilon - 1)},$$

$$\delta_{6A}^{(0)} = \frac{(\epsilon - 6)(\epsilon - 4)(\epsilon - 2)^2}{768(\epsilon - 3)^2}, \quad \delta_{6B}^{(0)} = \frac{(\epsilon - 6)(\epsilon - 5)(\epsilon - 4)(\epsilon - 2)^3}{384(\epsilon - 1)^2}. \quad (78)$$

Appendix D. Renormalization group equations at $n = 4$

For convenience we present the solution [4, 14] of the $n = 4$ renormalization group equations (40). The first two equations may be solved immediately.

$$\begin{aligned} \rho(t) &= \frac{\rho_0}{1 + b^2 \rho_0 t}, & \rho_0 &= \rho(0), & b^2 &= \frac{196}{45(4\pi)^2}; \\ \lambda(t) &= \frac{\lambda_0}{1 + a^2 \lambda_0 t}, & \lambda_0 &= \lambda(0), & a^2 &= \frac{133}{10(4\pi)^2}. \end{aligned} \quad (79)$$

In the UV limit $t \rightarrow \infty$ the solutions (79) manifest the asymptotic freedom for the two parameters. Indeed, the term “asymptotic freedom” is proper only for λ , which is a parameter of the loop expansion of theory (1). The positivity in the theory is consistent with $\lambda > 0$ and therefore, according to (79), we need $\lambda_0 > 0$ too. In contrast, ρ^{-1} is the coefficient of the topological Gauss-Bonnet term, which does not correspond to any gravitons interaction in the $n = 4$ case. The third equation (40) can be rewritten in terms of a new variable $w(t) = \xi(t)/\lambda(t)$

$$\left(t + \frac{1}{a^2 \lambda_0}\right) \frac{dw}{dt} = \frac{C}{a^2} (w^2 - kw + l) = \frac{C}{a^2} (w - w_1)(w - w_2), \quad (80)$$

where

$$C = \frac{10}{(4\pi)^2}, \quad k = \frac{183}{133}, \quad l = \frac{25}{133 \cdot 18}. \quad w_{1/2} = \frac{k}{2} \pm \sqrt{\frac{k^2}{4} - l}. \quad (81)$$

Let us notice that $w_1 \approx k \gg l/k \approx w_2 > 0$. The solution of Eq. (80) can be easily found and written in terms of the original variables, notations (81) and $w_0 = w(0)$, as

$$w = \frac{w_1 - X w_2}{1 - X}, \quad \text{where} \quad X = \left(\frac{w_0 - w_1}{w_0 - w_2}\right) \cdot (1 + \lambda_0 a^2 t)^m \quad (82)$$

and

$$m = \frac{C(w_1 - w_2)}{a^2} \approx 0.517.$$

The UV behavior of $w(t)$ depends on w_0 .

- i) For $w_0 < w_2$ the UV limit is $w \rightarrow w_2$;
- ii) $w_0 = w_{1,2}$ corresponds to the fixed points $w \equiv w_{1,2}$;
- iii) For $w_2 < w_0 < w_1$ the UV limit is $w \rightarrow w_2$, hence w_2 is a stable fixed point of the theory;
- iv) For $w_0 > w_1$ the UV limit is singular. The singularity is achieved at $X = 1$, that corresponds to the value

$$t_s = \frac{1}{\lambda_0 a^2} \left[\left(\frac{w_0 - w_1}{w_0 - w_2}\right)^{1/m} - 1 \right]. \quad (83)$$

For w_0 comparable to w_1 $t_s \gg 10\lambda_0^{-1}$. We remark that the applicability of the perturbative approach requires small λ_0 and therefore the singularity occurs at very high energies. For $w_0 \gg w_1$ the position of the singularity point t_s may be closer to zero. Since the value t_s is finite, this point is singular also for ξ .

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